Euler's equation: $ax^2y''+bxy'+cy=0 \quad x>0$

Characteristic Equation: $ar(r-1)+br+c=0$

Characteristic Roots: r_1, r_2

Case I: $r_1 \neq r_2$ real.

General Solution: $y = C_1 x^{r_1} + C_2 x^{r_2}$

Example: $(x+1)^2 y'' + 3(x+1)y' + 0.75y = 0 \quad x > -1$

Put in $y=(x+1)^r$ to get the same characteristic equation

$$r(r-1)+3r+0.75=0 \Rightarrow r^2+2r+0.75=0$$

$$\Rightarrow r_1 = -0.5,\ r_2 = -1.5$$

$$y = C_1(x+1)^{-0.5} + C_2(x+1)^{-1.5}$$

Case II: $r_1 \neq r_2$ complex $r_1 = \alpha + i\beta,\ r_2 = \alpha - i\beta.$

Complex solution: $\tilde{y} = x^{\alpha+i\beta}$

$$= x^{\alpha} \cdot x^{i\beta}$$

Recall: $t^k = (e^{\ln t})^k = e^{k\ln t} \quad t>0.$

$$\tilde{y} = x^{\alpha} \cdot x^{i\beta} = x^{\alpha} \cdot e^{i\beta \ln x}$$

$$= x^{\alpha}\left(\cos(\beta \ln x) + i\sin(\beta \ln x)\right)$$

$$= x^{\alpha}\cos(\beta \ln x) + i\,x^{\alpha}\sin(\beta \ln x)$$

$z = a+ib$

$\operatorname{Re} z = a,\ \operatorname{Im} z = b$

By the fact $\mathrm{Re}\,\tilde{y}$ and $\mathrm{Im}\,\tilde{y}$ are sol'ns, we know

$$x^{\alpha}\cos(\beta \ln x), \quad x^{\alpha}\sin(\beta \ln x)$$

are real sol'ns. Easy to see $W(x^{\alpha}\cos(\beta\ln x), x^{\alpha}\sin(\beta\ln x)) \neq 0$

Gen. sol'n: $y = C_1 x^{\alpha}\cos(\beta \ln x) + C_2 x^{\alpha}\sin(\beta\ln x)$

Example: $x^2 y'' + 3xy' + 5y = 0. \quad x>0$

Char. eqn: $r(r-1)+3r+5=0 \Rightarrow r^2+2r+5=0.$

$$\Rightarrow r^2+2r+1 = -4 \Rightarrow (r+1)^2 = -4 \Rightarrow r+1 = \pm 2i$$

Char. roots: $r = -1 \pm 2i.$

General sol'n: $y = C_1 x^{-1}\cos(2\ln x) + C_2 x^{-1}\sin(2\ln x)$

Case III: $r_1 = r_2 = r$ (automatically real)

We know one solution $y_1 = x^r$. To find another sol'n, we apply variation of parameter (reduction of order)

First notice that the ODE in this case should be

$$ax^2y'' + a(1-2r)xy' + ar^2 y = 0$$

Char. eqn: $aR(R-1) + bR + c = a(R-r)^2$

$$aR^2 + (b-a)R + c = aR^2 - 2arR + ar^2$$

$$\Rightarrow a=a, \quad b-a = 2ar, \quad c = ar^2 \Rightarrow b = (2r+1)a, \quad c = ar^2$$

So the standard form would be

$$y'' + \frac{1-2r}{x} y' + \frac{r^2}{x^2} y = 0$$

Set $y_2 = u(x) y_1(x) = u(x) \cdot x^r$, then $u(x)$ should satisfy

$$x^r u'' + \left(2r x^{r-1} + \frac{1-2r}{x} \cdot x^r\right) u' = 0$$

$$x^r u'' + \left(2r x^{r-1} + (1-2r) x^{r-1}\right) u' = 0$$

$$x^r u'' + x^{r-1} u' = 0$$

$$u'' + x^{-1} u' = 0 \Rightarrow \frac{u''}{u'} = -x^{-1}$$

Recall: y_1 sol'n for $y'' + py' + qy = 0$. Set $y_2 = u y_1$, then u satisfies

$$y_1 u'' + (2y_1' + p y_1) u' = 0$$

Integrate: $\ln|u'| = -\ln|x| = \ln|x|^{-1}$

$$\Rightarrow u' = \frac{1}{x} \Rightarrow u = \ln x$$

$y_2 = x^r \ln x$ is another solution.

Easy to see $W(x^r, x^r \ln x) \neq 0$.

General Solution: $y = C_1 x^r + C_2 x^r \ln x$

Recall: Principle of superposition: $y'' + py' + qy = 0$. if y_1, y_2 are solutions to the ODE with $W(y_1, y_2) \neq 0$, then the general solution is $y = C_1 y_1 + C_2 y_2$

Example: $x^2 y'' - 3xy' + 4y = 0$, $x > 0$

Char. eqn.: $r(r-1) - 3r + 4 = 0 \Rightarrow r^2 - 4r + 4 = 0 \Rightarrow r = 2, 2$

Gen. sol'n: $y = C_1 x^2 + C_2 x^2 \ln x$.

Formulas you should memorize so far:

(1) For $ay''+by'+cy=0$ a, b, c are real number

char. eqn. $ar^2+br+c=0 \Rightarrow$ char. roots r_1, r_2

Cases	General solution
$r_1 \neq r_2$ real	$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$
$r_1 \neq r_2$ complex	$y = C_1 e^{\alpha t}\sin\beta t + C_2 e^{\alpha t}\cos\beta t$ $\quad r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$
$r_1 = r_2 = r$ repeated	$y = C_1 e^{rt} + C_2 t e^{rt}$.

(2) For $ax^2y''+bxy'+cy=0$ a, b, c real numbers $x>0$

Char. eqn. $\Rightarrow$ char. roots r_1, r_2

Cases	General solution
$r_1 \neq r_2$ real	$y = C_1 x^{r_1} + C_2 x^{r_2}$.
$r_1 \neq r_2$ complex	$y = C_1 x^{\alpha}\cos(\beta \ln x) + C_2 x^{\alpha}\sin(\beta \ln x)$ $\quad r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$
$r_1 = r_2 = r$ repeated	$y = C_1 x^r + C_2 x^r \ln x$.

Remark: For the Euler's equation with $x<0$, the general solution would be easy: Just change all x into $|x|$. The IVP would be hard because of derivatives of absolute values.

Example: $x^2y'' - xy' + y = 0$. $y(-1) = 1$, $y'(-1) = 2$

The general solution is easy to obtain:

$$r(r-1) - r + 1 = r^2 - 2r + 1 = (r-1)^2 = 0 \Rightarrow r = 1, 1$$

$$y = C_1|x| + C_2|x|\ln|x|$$

But to determine C_1 and C_2, you have to take care of the absolute values.

Way 1: Keep the absolute value:

$$y(-1) = 1 \Rightarrow C_1 = 1$$

$$y'(x) = -C_1 + C_2\left(-\ln|x| + |x|\cdot\frac{1}{x}\right)$$

$$y'(-1) = -C_1 + C_2\left(0 + \frac{1}{-1}\right)$$

$$= -1 - C_2 = 2 \Rightarrow C_2 = -3$$

Solution: $y = |x| - 3|x|\ln|x|$

Caution:

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

$$(|x|)' = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$

But $(\ln|x|)' = \frac{1}{x}$ regardless of sign.

Way 2: Get rid of absolute value.

Since $x < 0$, $y = C_1 x + C_2 x\ln|x|$

$$y(-1) = 1 \Rightarrow C_1 = -1$$

$$y'(x) = C_1 + C_2\ln|x| + C_2\cdot x\cdot\frac{1}{x}$$

$$y'(-1) = -1 + C_2\cdot 0 + C_2\cdot 1 = 2$$

$$\Rightarrow C_2 = 3$$

Solution: $y = -x + 3x\ln|x|$.

It's indeed $y = C_1(-x) + C_2(-x)\ln|x|$ where $-$ sign is swallowed by C_1, C_2

Second order nonhomogeneous linear ODE

$$y'' + p(t)y' + q(t)y = g(t)$$

Structure theorem of solution: The general solution of such ODE looks like

$$y = y_c + Y = C_1 y_1 + C_2 y_2 + Y$$

where $y_c = C_1 y_1 + C_2 y_2$ is the general solution to the corresponding homogeneous ODE $y'' + p(t)y' + q(t)y = 0$

and Y is a particular solution to the nonhomogeneous ODE

$$y'' + p(t)y' + q(t)y = g(t)$$

Proof: Note the following fact: if Y_1, Y_2 are solutions of the nonhomog. ODE $y'' + py' + qy = g$. then $Y_1 - Y_2$ is a solution of the homog. ODE $y'' + py' + qy = 0$. In fact, Y_1, Y_2 being sol'ns means:

$$Y_1'' + pY_1' + qY_1 = g$$

$$Y_2'' + pY_2' + qY_2 = g$$

substract: $$Y_1'' - Y_2'' + p(Y_1' - Y_2') + q(Y_1 - Y_2) = 0.$$

$$(Y_1 - Y_2)'' + p(Y_1 - Y_2)' + q(Y_1 - Y_2) = 0.$$

Based on fact, if we know one particular solution Y to the nonhomog. ODE. then for any number C_1. C_2.

$$C_1 y_1 + C_2 y_2 + Y$$

will be a solution to the nonhomog. ODE. This can also be seen by direct verification:

$$(C_1y_1+C_2y_2+Y)''+p(C_1y_1+C_2y_2+Y)'+q(C_1y_1+C_2y_2+Y)$$
$$= (C_1y_1+C_2y_2)''+p(C_1y_1+C_2y_2)'+q(C_1y_1+C_2y_2)$$
$$+ Y''+pY'+qY$$
$$= 0 + g$$

By existence & uniqueness theorem, the gen. sol'n of a second order ODE involves at most two arbitrary variables. and we already got those two variables. So the theorem is proved.

$y_c = C_1y_1 + C_2y_2$ refered as the complementary solution

Y is referred as a particular solution

Gen. sol'n = Comp. sol'n + Parti. sol'n.

Second order nonhomogeneous linear ODE w/ const. coefficients

$$ay'' + by' + cy = g(t)$$

We know very well how to find the complementary solution. For the next part we'll focus on how to find a particular solution.

Way 1: Method of undetermined coefficients

Convenient but limited. Easy to generalize to higher order

Way 2: Variation of parameter.

Works for all cases but inconvenient. Higher order version way too complicated

Know $y_c = C_1 y_1 + C_2 y_2$.

$$Y = y_1 \int \frac{-y_2 \cdot g}{W(y_1, y_2)}\, dt + y_2 \cdot \int \frac{y_1 g}{W(y_1, y_2)}\, dt.$$

Method of undetermined coefficients.

Example: $y'' - 2y' - 3y = 3$

Idea: RHS is constant number. Try constant functions.

Complementary solution: $y_c = C_1 e^{-t} + C_2 e^{3t}$.

Set $Y = A$ to be a parti. sol'n.

$$Y'' - 2Y' - 3Y = -3A$$

Set it equal to the RHS $\Rightarrow -3A = 3 \Rightarrow A = -1$

$Y = -1$ is a particular sol'n

$\Rightarrow$ Gen. sol'n: $y = C_1 e^{-t} + C_2 e^{3t} - 1$.

Example: $y'' - y' - 2y = t^2 + 1$

Idea: RHS is a polynomial. Try polynomial function.

Comp. sol'n: $y_c = C_1 e^{2t} + C_2 e^{-t}$.

Set $Y = At^2 + Bt + C$ a parti. sol'n

Compute $Y'' - Y' - 2Y$

$$Y' = 2At + B. \quad Y'' = 2A.$$

$$Y'' - Y' - 2Y = 2A - (2At + B) - 2(At^2 + Bt + C)$$

$$= -2At^2 + (-2A - 2B)t + 2A - B - 2C.$$

Set it equal to RHS. $= t^2 + 1$ $(= t^2 + 0t + 1)$

$\Rightarrow -2A = 1, \; -2A - 2B = 0, \; 2A - B - 2C = 1$

$\Rightarrow A = -\frac{1}{2}, \; B = -A = \frac{1}{2}, \; C = \frac{1}{2}(2A - B - 1) = \frac{1}{2}(-1 - \frac{1}{2} - 1) = -\frac{5}{4}$

$$Y = -\frac{1}{2}t^2 + \frac{1}{2}t - \frac{5}{4}$$

Gen. sol'n: $y = C_1 e^{2t} + C_2 e^{-t} - \frac{1}{2}t^2 + \frac{1}{2}t - \frac{5}{4}$.

Att. Quiz: Find the general solution to

$$y'' - 5y' + 6y = t + 6.$$

$y'' + py' + qy = 0$

y_1 sol'n. $y_2 = uy_1$

$u(x)$ should satisfy:

$y_1 u'' + (2y_1' + py_1)u' = 0$

$$x^r u'' + \left(2 \cdot r x^{r-1} + \frac{1-2r}{x} \cdot x^r\right) u' = 0$$

$(ax^2 y'' + (1-2r)axy' + ar^2 y = 0$ b/c $aR(R-1) + bR + c = a(R-r)^2$

$aR^2 + (b-a)R + c = a(R^2 - 2rR + r^2)$

$b - a = -2ar \quad c = ar^2$

Std. form: $y'' + \frac{1-2r}{x} y' + \frac{r^2}{x^2} y = 0)$

Lecture Notes of Differential Equation